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DOPPLER FACTORS IN SATELLITE-TO-SATELLITE TRACKING

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APRIL 1974



GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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ABSTRACT

The Doppler factors occurring in range-rate satellite-to-satellite tracking measurements are derived with special relativistic effects included. The error resulting from the use of simplified expressions for these factors is discussed.

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DOPPLER FACTORS IN SATELLITE-TO-SATELLITE TRACKING

INTRODUCTION

The Doppler factors arising in one-way and two-way satellite tracking are well-known [1, 2]. The factors applicable to the satellite-to-satellite tracking configuration are not so readily available, however, and a special relativistic derivation of these factors is documented in this report.

THE DOPPLER EFFECT FORMULA

The relativistic formula for the Doppler effect [3] is often given in the form

$$f = f_0 \frac{\sqrt{1 - v^2/c^2}}{1 - \hat{e} \cdot \bar{v}/c} \quad (1)$$

In (1) f_0 is the proper frequency of the source (i.e. the frequency measured in the inertial system in which the source is at rest). The source itself is moving at a vector velocity \bar{v} with respect to an observer. The frequency detected by a receiver at rest in the inertial system of the observer is f . The vector \hat{e} is a unit vector in the direction of propagation as measured in the coordinate system of the observer, v is the magnitude of the vector \bar{v} , and c is the speed of light.

If the distance between the vehicle and the observer at the time of emission of the signal is r , then (letting the vector position of the vehicle be \bar{r} , differentiating $\bar{r} \cdot \bar{r} = r^2$ with respect to time, and setting $\dot{\bar{r}} = -r \hat{e}$)

$$\hat{e} \cdot \bar{v} = - \frac{dr}{dt} = -\dot{r} \quad (2)$$

and Equation (1) provides the downlink Doppler factor

$$f/f_0 = \frac{\sqrt{1 - v^2/c^2}}{1 + \dot{r}/c} \quad (3)$$

that holds good for one-way range-rate tracking, the Doppler factor being here

defined as the factor by which the transmitted frequency is multiplied to give the received frequency. To obtain the Doppler factors applicable in satellite-to-satellite tracking, however, a more general form of the Doppler formula (1) is required; one in which not only the transmitter but also the receiver of the signal is in motion with respect to the observer.

GENERAL DOPPLER EFFECT FORMULA

The required generalization of the Doppler formula (1) can be obtained from the latter by use of standard transformations of special relativity. Consider an observer at rest in an inertial frame Σ , a transmitter moving with vector velocity \bar{u} in Σ , and a receiver imbedded in a frame Σ' that moves with vector velocity \bar{v} with respect to the observer. The configuration is depicted in Figure 1. As

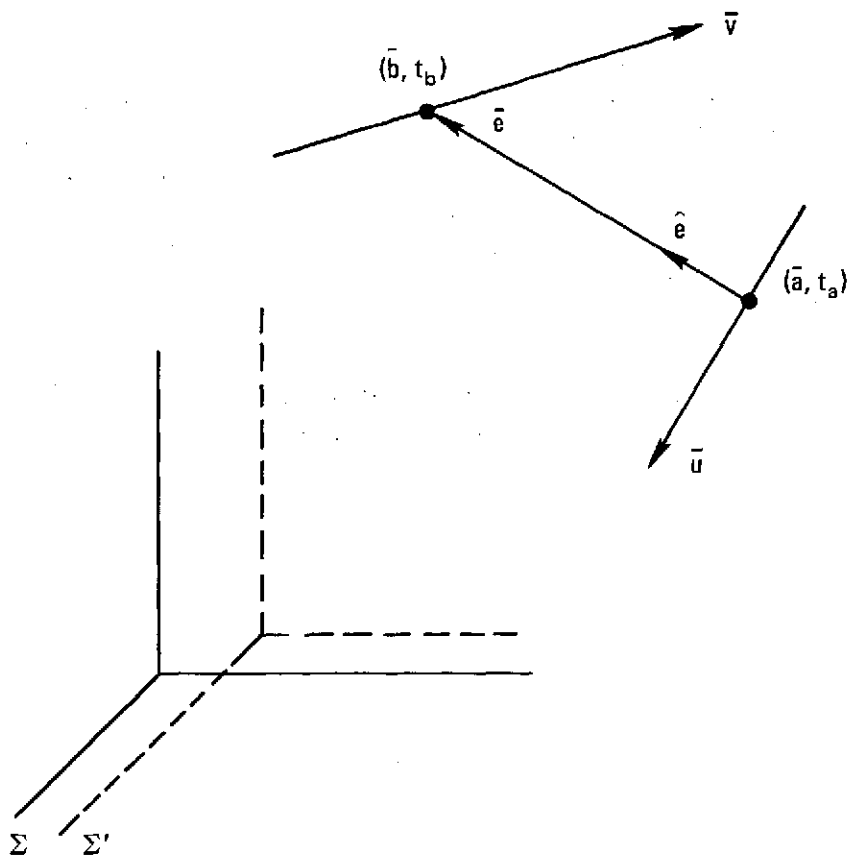


Figure 1. Doppler Effect Geometry

seen by the observer in Σ , let \bar{e} be the displacement vector drawn from the point \bar{a} where the signal is emitted at time t_a to the point \bar{b} where the signal is received at time t_b . Then the unit vector \hat{e} in the direction of propagation is

$$\hat{e} = \bar{e}/e \quad (4)$$

where e is the magnitude of \bar{e} and is equal to

$$e = c(t_b - t_a) \quad (5)$$

Following convention, the quantities above, being measured by the observer in Σ , are unprimed. The same quantities, as measured by an observer in Σ' , will be identified by a prime. With this notation, the Doppler Equation (1), as it applies to an observer in Σ' becomes

$$f' = f_0 \frac{\sqrt{1 - u'^2/c^2}}{1 - \hat{e}' \cdot \bar{u}'/c} \quad (6)$$

Here u' is the magnitude in Σ' of the vector velocity \bar{u}' of the transmitter.

To obtain the desired form of the Doppler equation, the quantities \bar{u}' , u' , and \hat{e}' in (6) must be expressed in terms of the unprimed quantities seen by the observer in Σ . The vector equation for the relativistic transformation of velocities is [3, Eqn. 2.55]

$$\bar{u}' = \frac{\sqrt{1 - v^2/c^2} \bar{u} + [(1 - \sqrt{1 - v^2/c^2}) \bar{u} \cdot \bar{v}/v^2 - 1] \bar{v}}{1 - \bar{u} \cdot \bar{v}/c^2} \quad (7)$$

Taking the scalar product of (7) with itself, and performing some algebraic manipulations, the well-known result [3, Eqn. 2.56]

$$\sqrt{1 - u'^2/c^2} = \frac{\sqrt{1 - v^2/c^2} \sqrt{1 - u^2/c^2}}{1 - \bar{u} \cdot \bar{v}/c^2} \quad (8)$$

is obtained.

To express \hat{e}' in terms of unprimed quantities, the Lorentz transformation in vector form [3, Eqn. 2.25] is used.

$$\bar{x}' = \bar{x} + \bar{v} \left[\frac{\bar{x} \cdot \bar{v}}{v^2} (1/\sqrt{1-v^2/c^2} - 1) - t/\sqrt{1-v^2/c^2} \right] \quad (9a)$$

$$t' = (t - \bar{x} \cdot \bar{v}/c^2) / \sqrt{1-v^2/c^2} \quad (9b)$$

Here (\bar{x}, t) are the coordinates of any given event. Substituting (\bar{b}, t_b) and (\bar{a}, t_a) successively into (9a), taking the difference of the two resulting equations, and employing (5)

$$\bar{e}' = \bar{e} + \bar{v} \left[\frac{\bar{e} \cdot \bar{v}}{v^2} (1/\sqrt{1-v^2/c^2} - 1) - e/(c\sqrt{1-v^2/c^2}) \right] \quad (10)$$

Substituting in the same way into (9b), and multiplying the resulting difference by c ,

$$e' = (e - \bar{v} \cdot \bar{e}/c) / \sqrt{1-v^2/c^2} \quad (11)$$

Taking the scalar product of (7) and (10), dividing both by (11) and by the constant c , and subtracting the result from unity, there obtains

$$1 - \hat{e}' \cdot \bar{u}'/c = \frac{(1 - \hat{e} \cdot \bar{u}/c)(1 - v^2/c^2)}{(1 - \hat{e} \cdot \bar{v}/c)(1 - \bar{u} \cdot \bar{v}/c^2)} \quad (12)$$

Substituting (8) and (12) into (6)

$$f' = f_0 \frac{(1 - \hat{e} \cdot \bar{v}/c) \sqrt{1-u^2/c^2}}{(1 - \hat{e} \cdot \bar{u}/c) \sqrt{1-v^2/c^2}} \quad (13)$$

Equation (13) is the desired generalized Doppler formula. It relates the frequency f_0 measured aboard a transmitting vehicle moving with vector velocity \bar{u} at the time of transmission of a signal, to the frequency received aboard a vehicle moving with vector velocity \bar{v} , at the time of reception of the signal the unit vector \hat{e} being directed from the point where the transmitter is located at the time of transmission to the point where the receiver is located at the time of reception. Using this equation all of the Doppler factors involved in satellite-to-satellite tracking can be written down immediately by suitably specifying the transmitter and receiver velocities.

Note, incidentally, that (13) may be written in the form

$$(f' \sqrt{1 - v^2/c^2}) = (f_0 \sqrt{1 - u^2/c^2}) \frac{1 - \hat{e} \cdot \bar{v}/c}{1 - \hat{e} \cdot \bar{u}/c} \quad (14)$$

from which it becomes apparent that the square roots that appear in (13) may be regarded as corrections to the classical Doppler factor $(1 - \hat{e} \cdot \bar{v}/c) / (1 - \hat{e} \cdot \bar{u}/c)$ [3, Eqn. 1.22] for the effect of relativistic time dilation on the frequencies measured aboard the vehicles.

SATELLITE-TO-SATELLITE TRACKING DOPPLER FACTORS

Referring to Figure 2, let a signal from a ground-based tracking station be received at time t_1 by a first satellite moving with a velocity \bar{v} . Let \bar{s} be the vector drawn from the tracking station to the point where the signal is received. After an interval corresponding to the transponder delay, a signal is sent from the first satellite at time t_2 to a second satellite where it is received at time t_3 . Let \bar{q} be the displacement vector drawn from the point of transmission at time t_2 to the point of reception at time t_3 . In a similar manner, let the round-trip of the signal be completed by retransmission back to the ground station along the vector paths \bar{p} and \bar{r} . Note, however, that the direction of the latter vectors has been taken opposite to the direction of propagation.

To obtain the uplink Doppler factor u_1 , the quantities \bar{u} and \bar{v} appearing in (13) are defined to be the vector velocities of the tracking station and the first satellite respectively. Thus in (13) $\bar{u} = 0$ and $\hat{e} = \bar{s}/s = \hat{s}$ giving

$$u_1 = \frac{1 - \dot{s}/c}{\sqrt{1 - v^2(s)/c^2}} \quad (15)$$

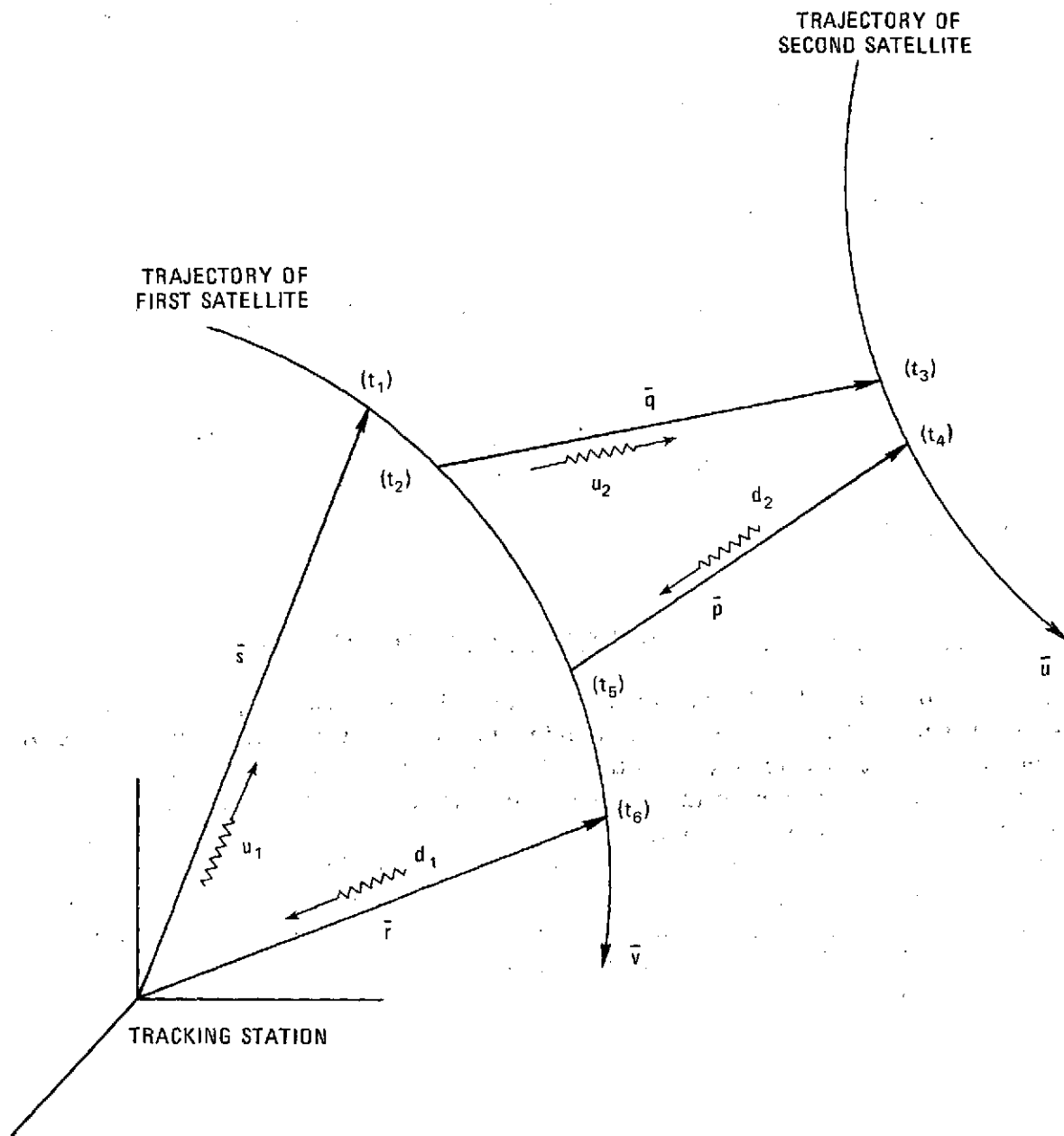


Figure 2. Satellite-to-Satellite Tracking Configuration

where $v(s)$ is the speed of the first satellite at the time t_1 when the signal was received at the vector position \bar{s} .

The downlink Doppler factor d_1 has already been given in (3)

$$d_1 = \frac{\sqrt{1 - v^2(r)/c^2}}{1 + \dot{r}/c} \quad (16)$$

The second uplink Doppler factor u_2 is obtained from (13) by interchanging \bar{u} and \bar{v}

$$u_2 = \frac{(1 - u_q/c) \sqrt{1 - v^2(q)/c^2}}{(1 - v_q/c) \sqrt{1 - u^2(q)/c^2}} \quad (17)$$

Here $u_q \equiv \hat{q} \cdot \bar{u}$ is the component of the velocity of the second satellite along the \hat{q} direction, and v_q is the like component of the first satellite.

The remaining Doppler factor d_2 follows from (13) by setting $\hat{e} = -\hat{p}$ giving

$$d_2 = \frac{(1 + v_p/c) \sqrt{1 - u^2(p)/c^2}}{(1 + u_p/c) \sqrt{1 - v^2(p)/c^2}} \quad (18)$$

In coherent round-trip Doppler tracking, the Doppler factors appear only as products, for example $u_1 d_1$ or $u_1 u_2 d_1 d_2$. In such cases, the square roots appearing in (15)-(18) cancel out provided that the speed of the vehicles does not change appreciably over the duration of the round-trip. Consequently negligible error is incurred in these cases if the classical Doppler factors

$$u_1 = 1 - \dot{s}/c \quad (19)$$

$$d_1 = 1/(1 + \dot{r}/c) \quad (20)$$

$$u_2 = \frac{1 - u_q/c}{1 - v_q/c} \quad (21)$$

$$d_2 = \frac{1 + v_p/c}{1 + u_p/c} \quad (22)$$

are used in place of (15)-(18).

APPROXIMATE DOPPLER FACTORS

If the first satellite in Figure 2 is geostationery, which is a case of practical interest [4], it is possible to further simplify (21) and (22) because of the relatively small magnitude of the velocity of the geostationary satellite. The denominator of (21) can be expanded to give

$$\begin{aligned} u_2 &= (1 - u_q/c) \left(1 + \frac{v_q}{c} + \frac{v_q^2}{c^2} \right) \\ &= 1 - \frac{(u_q - v_q)}{c} \left(1 + \frac{v_q}{c} \right) \\ &= 1 - \frac{\dot{q}}{c} (1 + v_q/c) \end{aligned} \quad (23)$$

since $u_q - v_q = \dot{q}$. Consequently the approximation

$$u_2 = 1 - \dot{q}/c \quad (24)$$

produces a relative error ϵ_r in range rate that is less than

$$\epsilon_r \leq \frac{v_q}{c} \leq v/c \quad (25)$$

and an absolute error less than

$$\epsilon_a \leq \dot{q} v / c \quad (26)$$

Typical values for u and v in the ATS/NIMBUS-F experiment are [5] 7.5 km/sec and 0.08 km/sec giving

$$\epsilon_a \leq 2.0 \text{ millimeters/sec} \quad (27)$$

An error of approximately the same magnitude but of opposite sign arises in using

$$d_2 = \frac{1}{1 + \dot{p}/c} \quad (28)$$

in place of (22).

An error of 2.0 mm/sec is not insignificant. However this error, being of opposite sign in (24) and (28), cancels out in round-trip Doppler measurements, in which only the products of uplink and downlink Doppler factors appear. A more complete discussion of the errors is given in the Appendix.

SUMMARY

Using the transformations of special relativity, the four Doppler factors (15), (16), (17), and (18) encountered in satellite-to-satellite tracking are derived. It is shown, in the case of round-trip tracking from the ground through a geostationary satellite to a close-in earth orbiting satellite and then back to the ground view the same path, that these Doppler factors may be replaced by the simpler ones given by (19), (20) and either (21) and (22) or (24) and (28).

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APPENDIX

APPROXIMATION ERRORS

A. NEGLECT OF RELATIVISTIC FACTORS

Using Equations (15) and (16) the quantity $c(1 - u_1 d_1)/2$ may be expanded in inverse powers of c to give

$$c(1 - u_1 d_1)/2 = \frac{\dot{r} + \dot{s}}{2} - \frac{\dot{r}(\dot{r} + \dot{s})}{2c} + \frac{v^2(r) - v^2(s)}{4c} + \dots \quad (A1)$$

where terms in negative powers of c other than the first power have been neglected. If Equations (19) and (20) are used instead of (15) and (16) to calculate $c(1 - u_1 d_1)/2$ the results

$$c(1 - u_1 d_1)/2 = \frac{\dot{r} + \dot{s}}{2} - \frac{\dot{r}(\dot{r} + \dot{s})}{2c} + \dots \quad (A2)$$

The first term on the right hand side of (A1) is the mean range-rate of the first satellite, and this term is much larger than the remaining ones. It follows that the use of the classical Equations (19) and (20) instead of relativistic (15) and (16) will result in an error in the calculation of range-rate of the magnitude

$$\epsilon_a = \frac{v^2(r) - v^2(s)}{4c} \quad (A3)$$

The numerator of (A3) represents the change in the square of the speed of the satellite during the time it travels from point \bar{s} to \bar{r} .

$$\epsilon_a = \frac{1}{4c} \Delta v^2 = \frac{1}{2c} v \frac{dv}{dt} \Delta t \quad (A4)$$

In the case of an earth-orbiting satellite in unpowered flight, dv/dt will not exceed the acceleration of gravity at the earth's surface (about 9.8 m/s^2).

In single satellite tracking the time interval in question is a transponder delay that might be, very roughly, about 1 microsecond. Using these values, and 7.5 km/s for v there results

$$\epsilon_a \leq 1.2 \times 10^{-10} \text{ m/sec} \quad (\text{A5})$$

which is clearly negligible. Like results will obtain in satellite-to-satellite tracking where the cancellation of relativistic factors in the product $u_1 u_2 d_1 d_2$ takes place between factors separated by successive transponder delays (for example, the radical in u_1 is cancelled not by the radical in d_1 but rather by one of the radicals in u_2).

B. USE OF SIMPLIFIED FACTORS

Expansion of the $c(1 - u_2 d_2)/2$ using (21) and (22) gives

$$c(1 - u_2 d_2)/2 = \frac{1}{2} (\dot{q} + \dot{p}) \left(1 - \frac{\dot{p}}{c}\right) - \frac{1}{2c} (v_p \dot{p} - v_q \dot{q}) \quad (\text{A6})$$

Using the approximate Doppler factors (24) and (28), instead of (21) and (22) yields

$$c(1 - u_2 d_2)/2 = \frac{1}{2} (\dot{q} + \dot{p}) \left(1 - \frac{\dot{p}}{c}\right) \quad (\text{A7})$$

The error incurred in using (24) and (28) to calculate the mean range rate $(\dot{q} + \dot{p})/2$ between the two satellites is therefore

$$\epsilon_a = (v_p \dot{p} - v_q \dot{q})/2c \quad (\text{A8})$$

which is half the difference between the errors (see 26) incurred in using (28) or (24) separately. The error here is the increment

where $\Delta \tau = t_4 - t_3$ is the transponder delay in the second satellite and $\Delta t = t_5 - t_2$ is given by

$$\Delta t = \frac{q + p}{c} + \Delta \tau \doteq 2q/c \quad (\text{A13})$$

The size of the change in velocity of the second satellite is less than

$$|\Delta \vec{u}| \leq g \Delta \tau \quad (\text{A14})$$

where $g = 9.8 \text{ m/s}^2$.

Substituting (A12) and (A14) into (A11)

$$\Delta u_q \leq g \Delta \tau + \frac{u^2}{q} \Delta \tau + \frac{u v}{q} \Delta t \quad (\text{A15})$$

Evaluating Δv_q in the same way

$$\Delta v_q \leq g \Delta t + \frac{v^2}{q} \Delta t + \frac{u v}{q} \Delta \tau \quad (\text{A16})$$

where here g may be taken to have a smaller value because of the remote location of the geostationary satellite. From (A15), (A16), a bound on ϵ_a , Equation (A9) may be determined. The contribution from the first term on the right hand side of (A16) is much larger than all the rest, giving the result

$$\epsilon_a \leq u g \Delta t / 2 c \quad (\text{A17})$$

Using $u = 7.5 \text{ km/s}$, $g = 0.2 \text{ m/s}^2$, and $\Delta t = 0.25 \text{ s}$

$$\epsilon_a < 6.2 \times 10^{-7} \text{ m/s}$$

which is negligible.

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